UDC 539.3 : 534.1

ON THE BUBNOV-GALERKIN METHOD IN THE NONLINEAR THEORY OF VIBRATIONS OF VISCOELASTIC SHELLS

PMM Vol. 37, №6, 1973, pp. 1117-1124

I. I. VOROVICH and L. P. LEBEDEV
(Rostov-on-Don)

(Received October 16, 1972)

Results in [1] are extended to the case of vibrations of shallow and nonshallow viscoelastic (and elastic) shells. A uniqueness theorem is proved in a somewhat broader class of functions than in [1].

1. General formulation of the problem. The fundamental notation used in [1] is presented below with slight modifications.

Let ω be the element in some complete separable Hilbert space H_1 with the scalar product $(\omega_1 \cdot \omega_2)$. A_1 is a linear unbounded operator given on some set E_1, compact everywhere in H_1, with the following properties:

1) A_1 is a symmetric positive-definite operator.

2) If $\omega \in E_1$, then $A_1 \omega \in H_1$.

The scalar product and the norm are introduced in E_1

$$(\omega_1 \cdot \omega_2)_2 = (A_1 \omega_1 \cdot \omega_2), \qquad \|\omega\|_2{}^2 = (\omega \cdot \omega)_2$$

The complement of E_1 in the norm $\|\cdot\|_2$ is the space H_2.

3) A_1 possesses the eigenvectors ψ_n forming a complete system of vectors in the space H_2.

$E_2(a, b)$ is a set of elements $\omega(t)$ dependent on the parameter t such that $\omega \in E_1$, $\omega_t \in H_1$ (*) for any $a \leqslant t \leqslant b$, and ω as an element of H_2 and ω_t as an element of H_1 are continuous functions of the parameter t in $[a, b]$.

$E_3(a, b)$ is a subset of elements from $E_2(a, b)$ representable as the finite sums $\Sigma d_k(t) \chi_k$, where $d_k(t) \in C^{(1)}(a, b)$, $\chi_k \in H_2$.

The closure of $E_2(a, b)$ in the norm

$$(\omega_1 \cdot \omega_2)_{3,a,b} = \int_a^b \{(\omega_{1t} \cdot \omega_{2t}) + (\omega_1 \cdot \omega_2)_2\} \, dt$$

is called the space $H_3(a, b)$.

4) If $\omega_n \to \omega_0$ weakly in $H_3(a, b)$, then $\omega_n \to \omega_0$ strongly in H_1 uniformly in all $a \leqslant t \leqslant b$.

It has been shown in [1] that $H_3(a, b)$ is a separable space and $E_3(a, b)$ is compact everywhere in $H_3(a, b)$.

D° is a subset of $H_3(0, T)$ formed by the closure of a subset of functions from $E_3(0, T)$ in the norm of $H_3(0, T)$ such that $d_k(T) = 0$.

An equation of the following kind is considered:

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*) The subscripts t, α_k denote differentiation with respect to t, α_k.

$$\omega_{tt} = -A_1\omega - A_2\omega - B^t(\omega, \omega) - K\omega_t + F(t) \tag{1.1}$$

with the initial conditions

$$\omega|_{t=0} = g, \qquad \omega_t|_{t=0} = h \tag{1.2}$$

Equation (1.1) differs from (1.10) in [1] by the term $B^t(a, \omega)$, which is a nonlinear operator of two variables and represents the effect of internal "viscous" forces.

The assumptions in [1], relative to the operators A_1, A_2, K, are presented below with slight modifications.

The relationship $A_1\omega + A_2\omega = \operatorname{grad}_{H_1}\Phi(\omega)$ holds on E_1, where Φ is a functional given on H_2 and

$$\Phi(\omega_0 + \omega_1) - \Phi(\omega_0) = (A_3\omega_0 \cdot A_4\omega_1) + \alpha(\omega_0, \omega_1) \tag{1.3}$$

The $\omega_0, \omega_1 \in H_2$ in (1.3) are arbitrary; the operator A_3 is nonlinear, A_4 is a bounded linear operator from H_2 in H_1; the functional $\alpha(\omega_0, \omega_1)$ is such that $\lim |\alpha| \|\omega_1\|_2^{-1} = 0$, if $\|\omega_1\|_2 \to 0$.

5) If $\omega \in H_3(a, b)$, then

$$0 \leqslant \int_a^b (K\omega_t \cdot \omega_t)\, dt < \infty, \qquad a < b.$$

6) Φ is a nonnegative functional in H_2 such that $\|\omega\|_2 \leqslant \varphi_1(r)$ follows from $\Phi(\omega) \leqslant r$. Here and henceforth $\varphi_k(r)$ are functions bounded in each finite segment of variation of r.

7) If $\omega_0, \omega_1 \in H_3(0, T)$, then $(A_3\omega_0 \cdot A_4\omega_1)$ is a function summable in $[0, T]$. If $\omega_n \to \omega_0$ weakly in $H_3(0, T)$, then

$$\lim_{n\to\infty} \int_0^T (A_3\omega_n \cdot A_4\omega_1)\, dt = \int_0^T (A_3\omega_0 \cdot A_4\omega_1)\, dt$$

$$\lim \int_0^T |\alpha|\, dt \, \|\omega_1\|_{3,0,T}^{-1} = 0, \qquad \text{if} \quad \|\omega_1\|_{3,0,T} \to 0$$

Additional conditions besides those used in [1] are required for the investigation of (1.1).

8) If $\omega \in E_3(a, b)$, then $\Phi(\omega)$ is summable in $[a, b]$.

The operator $B^t(a, \omega)$ with the domain of definition $a \in E_2(0, T)$, $\omega \in E_1$ has the form $B^t(a, \omega) = \operatorname{grad}_{H_1(\omega)}(C^t a \cdot D\omega)$.

Here C^t, D are nonlinear operators acting in the space H_1 from $H_3(0, t)$ and H_2, respectively, for all $0 \leqslant t \leqslant T$ Moreover, for any $a \in H_3(0, t)$ and ω_0, $\omega_1 \in H_2$ the following relationship is valid:

$$(C^t a \cdot D(\omega_0 + \omega_1)) - (C^t a \cdot D\omega_0) = (C^t a \cdot A_5(\omega_0, \omega_1)) + \beta(a, \omega_0, \omega_1)$$

where the operator $A_5(\omega_0, \omega_1)$ is linear and bounded in the variable ω_1 from H_2 into H_1, where $\lim |\beta| \|\omega_1\|_2^{-1} = 0$, if $\|\omega_1\|_2 \to 0$.

9) The time segment $[0, T]$ can be separated into $n(T)$ parts $0 = t_0 < t_1 < \ldots < t_n = T$ so that the operator C^t has the form $C^t a = C_1^t a + \ldots + C_n^t a$.

Here the operator $C_k^t\, a$ depends only on the values of the element $a \in H_3(0, T)$ on the segment $[t_{k-1}, t_k]$; $C_k^t\, a \equiv 0$, if $t < t_{k-1}$. Moreover, for any element $\omega \in E_3(0, t_k)$ the following inequalities are satisfied:

$$\left|\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t}\left(C_k^{\tau}\omega\cdot\frac{\partial D\omega(\tau)}{\partial\tau}\right)d\tau dt\right|\leqslant\frac{1}{2}\int_{t_{k-1}}^{t_k}\Phi(\omega)\,d\tau \tag{1.4}$$

$$\left|\int_{t_{k-1}}^{t_k}\int_{0}^{t}\sum_{j=1}^{k}\left(C_j^{\tau}\omega\cdot\frac{\partial D\omega}{\partial\tau}\right)d\tau dt\right|\leqslant\varphi_2(\|\omega\|_{3,0,t_{k-1}})\left\{\int_0^{t_k}\Phi(\omega)\,dt\right\}^{1/2}$$

10) If ω, $a\in H_3(0,\ T)$, then $(C^t\omega\cdot A_5(\omega, a))$ is a function summable on $[0,\ T]$ bounded uniformly if $\|\omega\|_2$, $\|a\|_2$ are bounded uniformly on $[0,\ T]$ and if $\omega_n\to\omega_0$ weakly in $H_3(0,\ T)$, then

$$\lim_{n\to\infty}\int_0^T(C^t\omega_n\cdot A_5(\omega_n, a))\,dt=\int_0^t C^t\omega_0\cdot A_5(\omega_0, a))\,dt$$

11) $$\lim\int_0^T|\beta(a,\omega_0,\omega_1)|\,dt\,\|\omega_1\|_{3,0,T}^{-1}=0,\quad \text{if}\quad \|\omega_1\|_{3,0,T}\to 0$$

12) For any arbitrary element $\omega\in E_3(0,\ T)$ for all $0\leqslant t\leqslant T$

$$\left|\int_0^t\left(C^{\tau}\omega\cdot\frac{\partial D\omega}{\partial\tau}\right)d\tau\right|\leqslant\frac{1}{2}\Phi(\omega(t))+\varphi_3\left(\int_0^t\Phi(\omega)\,d\tau\right)$$

As in [1], the concept of the generalized solution is introduced.

Definition 1.1. An element $\omega\in H_3(0,\ T)$, satisfying the integral relation

$$\int_0^T\{-(\omega_t\cdot\omega_{1t})+(A_3\omega\cdot A_4\omega_1)+(C^t\omega\cdot A_5(\omega,\omega_1))+$$
$$(K\omega_t\cdot\omega_1)-(F\cdot\omega_1)\}\,dt-(h\cdot\omega_1)|_{t=0}=0$$

and the first of the initial conditions (1.2) in the following sense:

$$\lim\|\omega-g\|_1=0\quad\text{for}\quad t\to 0 \tag{1.5}$$

for arbitrary $\omega_1\in D^\circ$ is called a generalized solution of (1.1) with the initial conditions (1.2).

The generalized solution is sought approximately by the Bubnov-Galerkin method in the system of differential equations

$$(\omega_{mtt}\cdot\chi_l)+(A_3\omega_m\cdot A_4\chi_l)+(C^t\omega_m\cdot A_5(\omega_m,\chi_l))+$$
$$(K\omega_{mt}\cdot\chi_l)-(F\cdot\chi_l)=0,\qquad l=1,\ldots,m \tag{1.6}$$

$$\omega_m=\sum_{l=1}^m q_{ml}(t)\,\chi_l$$

with the initial conditions

$$q_{ml}(0)=(g\cdot\chi_l),\qquad q_{ml}^{\cdot}(0)=(h\cdot\chi_l) \tag{1.7}$$

Here χ_l is some complete system in H_2, which is considered orthonormalized in H_1 for convenience.

Theorem 1.1. Let Conditions (1)–(12) be satisfied. In this case (1.1) with the initial conditions (1.2) has at least one generalized solution in the sense of Definition 1.1 on the segment $[0, T]$ for arbitrary T, provided that

$$\mathbf{h} \in H_1, \quad \mathbf{g} \in H_2, \quad \int_0^T \|\mathbf{F}\|_1^2 \, dt < \infty$$

As in [1], Theorem 1.1 results from the following theorem.

Theorem 1.2. Let all the conditions of Theorem 1.1 be satisfied and let χ_l be some system complete in H_2 and orthonormal in H_1. In this case the system of differential equations (1.6) with the initial conditions (1.7) has at least one solution on the whole segment $[0, T]$ for each m. The set of approximate solutions ω_m is weakly compact in $H_3(0, T)$ and contains an infinite subset ω_m, each of whose limit points is a generalized solution of (1.1) in the sense of the Definition 1.1.

Structurally, the proof of Theorem 1.2 agrees with the proof of Theorem II in [1]: the system (1.6), (1.7) is reduced to an operator equation with a completely continuous operator, and then the Schauder fixed point theorem of a completely continuous operator is used. The existence of a solution of (1.6), (1.7) in some finite time segment $[0, T_1]$ is proved by such a method, and is then extended to the whole segment $[0, T]$ in a finite number of steps. From a priori estimates of the solution of the system there results that the sequence of approximate solutions is a weakly compact set in $H_3(0, T)$. Using Conditions (7) and (10), as in [1], it can be shown that each weak limit of this set is a generalized solution of the problem.

The main part of the whole proof is to obtain the following a priori estimates:

$$\|\omega_{mt}\|_1 \leqslant m_1, \quad \|\omega_m\|_2 \leqslant m_2, \quad \|\omega_m\|_{3,0,T} \leqslant m_3 \tag{1.8}$$

Here and henceforth m_k are some positive constants.

The method of mathematical induction is used for the proof: considering the estimates satisfied on the segment $[0, t_{k-1}]$, they must be extended to the segment $[0, t_k]$. The proof of the estimates on the segment $[0, t_1]$ is obtained from the proof on $[0, t_k]$ for $k = 1$.

The system (1.6) can be written as follows:

$$q_{ml}^{\cdot\cdot} = -\frac{\partial\Phi(\omega_m)}{\partial q_{ml}} - \left(\mathbf{C}^t\omega_m \cdot \frac{\partial \mathbf{D}\omega_m}{\partial q_{ml}}\right) - (\mathbf{K}\omega_{mt}\cdot\chi_l) + (\mathbf{F}\cdot\chi_l) \tag{1.9}$$
$$l = 1, \ldots, m$$

The lth equation in (1.9) is multiplied by q_{ml}, the equalities obtained are added and integrated with respect to time between the limits 0 and t, and then with respect to the parameter t between 0 and t_k. The part dependent on the values of ω_m on the segment $[t_{k-1}, t_k]$ is extracted from this equality and the Condition (5) is taken into account. Consequently

$$\int_{t_{k-1}}^{t_k} \{\|\omega_{mt}\|_1^2 + 2\Phi(\omega_m)\}\, dt + 2\int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t} \left(\mathbf{C}_k^\tau\omega_m \cdot \frac{\partial \mathbf{D}\omega_m}{\partial\tau}\right) d\tau dt \leqslant$$
$$L(\omega_m) + \int_{t_{k-1}}^{t_k}\int_{t_{k-1}}^{t} (\mathbf{F}\cdot\omega_{m\tau})\, d\tau dt - 2\int_{t_{k-1}}^{t_k}\int_0^t \sum_{i=1}^{k-1}\left(\mathbf{C}_i^\tau\omega_m \cdot \frac{\partial \mathbf{D}\omega_m}{\partial\tau}\right) d\tau dt$$

Values of ω_m only on the segment $[0, t_{k-1}]$ are present in $L(\omega_m)$. Using Condition

(9), the estimate

$$\int_{t_{k-1}}^{t_k} \Phi(\omega_m(\tau))\, d\tau \leqslant m_4 \tag{1.10}$$

can be deduced from the inequality obtained above.

To prove the estimates (1.8), all the manipulations carried out above are repeated, except the second integration with respect to the parameter t. These estimates are derived, as in [1], from the inequality obtained by such a method taking account of the already proved inequality (1.10) and Conditions (9), (12).

2. Problem of shell vibrations. The following version of the nonlinear theory of viscoelastic shells will be considered [2]:

$$\varepsilon_{11} = e_{11} + \frac{1}{2}\psi_1^2 = \frac{u_{1\alpha_1}}{A_1} + \frac{A_{1\alpha_2}u_2}{A_1A_2} + k_{11}u_3 + \frac{1}{2}\psi_1^2 \quad (1 \rightleftarrows 2) \tag{2.1}$$

$$2\varepsilon_{12} = 2e_{12} + \psi_1\psi_2 = \frac{A_1}{A_2}\left(\frac{u_1}{A_1}\right)_{\alpha_2} + \frac{A_2}{A_1}\left(\frac{u_2}{A_2}\right)_{\alpha_1} - 2k_{12}u_3 + \psi_1\psi_2$$

$$\varkappa_{11} = -A_1^{-1}\psi_{1\alpha_1} - A_{1\alpha_2}\psi_2(A_1A_2)^{-1} \quad (1 \rightleftarrows 2)$$

$$2\varkappa_{12} = -A_1A_2^{-1}(\psi_1A_1^{-1})_{\alpha_2} - A_2A_1^{-1}(\psi_2A_2^{-1})_{\alpha_1}$$

$$T_{ij} = T_{ijy} + T_{ijB} = E_{ijkl}\varepsilon_{kl} + \int_0^t C_{ijkl}(t,\tau)\,\varepsilon_{kl}(\tau)\, d\tau$$

$$M_{ij} = M_{ijy} + M_{ijB} = D_{ijkl}\varkappa_{kl} + \int_0^t B_{ijkl}(t,\tau)\,\varkappa_{kl}(\tau)\, d\tau$$

The following notation is used here: $\omega = (u_1, u_2, u_3)$ is the displacements of points of the shell middle surface S^* with the internal coordinates α_1, α_2; A_i^2, $2C = 0$ are coefficients of the first quadratic form of the surface S^*; k_{ij} are the curvatures of S^*; ε_{ij} are the tension and shear strains; $\varkappa_{ij}$ are the curvature changes; ψ_i are the turning angles of the coordinate lines; T_{ij} are the shear stresses, M_{ij} are the moments; $2h\ (\alpha_1, \alpha_2)$ is the shell thickness; $E_{ijkl}, C_{ijkl}, D_{ijkl}, B_{ijkl}$ are the shell elastic and viscous characteristics

$$E_{ijkl} = E_{klij}, \qquad C_{ijkl} = C_{klij}, \quad D_{ijkl} = {}^1/_3 h^2 E_{ijkl}, \qquad B_{ijkl} = {}^1/_3 h^2 C_{ijkl}$$

In the "shallow" theory case (V. Z. Vlasov version) $\psi_1 = A_1^{-1}u_{3\alpha_1}$ $(1 \rightleftarrows 2)$. In the "nonshallow" theory case $\psi_1 = A_1^{-1}u_{3\alpha_1} - k_{11}u_1$ $(1 \rightleftarrows 2)\ (k_{12} = k_{21} = 0)$.

The Hamilton-Ostrogradskii principle dictates the following definition of the generalized solution:

$$\int_0^T \int_\Omega \{T_{ij}\delta\varepsilon_{ij} + M_{ij}\delta\varkappa_{ij}\}\, A_1A_2\, d\alpha_1 d\alpha_2 dt = \int_0^T \int_\Omega \{F_i\delta u_i + 2\rho h u_{it}\delta u_{it}\} \times A_1A_2\, d\alpha_1 d\alpha_2 dt + \int_\Omega h_i \delta u_i A_1A_2\, d\alpha_1 d\alpha_2 \Big|_{t=0} \tag{2.2}$$

if the boundary conditions are

$$\omega|_\Gamma = 0, \qquad \frac{\partial u_3}{\partial n}\Big|_\Gamma = 0 \tag{2.3}$$

Here $\delta\varepsilon_{ij} = \delta e_{ij} + {}^1/_2(\psi_i\delta\psi_j + \psi_j\delta\psi_i)$; Ω is the domain with boundary Γ occupied by the shell planform; the variation sign δ means that the possible displacement $\delta\omega$ which is considered zero for $t = T$ must be substituted in place of the vector function ω; F_i is the distributed load; ρ is the density.

The initial conditions are the following:

$$u_i|_{t=0} = g_i, \qquad u_{it}|_{t=0} = h_i \tag{2.4}$$

Let the following conditions be satisfied:

a) Ω is a connected bounded domain which is a finite sum of star domains; its boundary Γ consists of a finite number of closed contours of the Liapunov class $Л_1(m, 0)$;

b) A_i, $A_{i_{\alpha_k}}$, k_{ij}, $k_{ij_{\alpha_l}}$, ρ, h are measurable functions bounded on Ω, where $0 < m_5 \leqslant A_i$, ρ, $h \leqslant m_6$;

c) The energy inequality $E_{ijkl}\varepsilon_{ij}\varepsilon_{kl} \geqslant m_7\varepsilon_{ij}\varepsilon_{ij}$, $m_7 > 0$ is satisfied for all symmetric tensors ε_{ij};

d) The functions

$$C_{ijkl}(t, \tau), \quad B_{ijkl}(t, \tau), \quad \frac{\partial}{\partial t} C_{ijkl}(t, \tau), \quad \frac{\partial}{\partial t} B_{ijkl}(t, \tau)$$

are measurable in the set of variables t, τ on the triangle $0 \leqslant \tau \leqslant t \leqslant T$, and for all $t \in [0, T]$ are summable in $[0, t]$ in the variable τ, and for all $\tau \in [0, T]$ are summable in $[\tau, T]$ in the variable t, where

$$\int_{t-\lambda}^{t} |C_{ijkl}(t, s)|\, ds + \int_{\tau}^{\tau+\lambda} \left|C_{ijkl}(s, \tau)\right| ds \leqslant \varphi_4(\lambda), \qquad |C_{ijkl}(t, t)| \leqslant m_8$$

$$\int_{t-\lambda}^{t} \left|\frac{\partial}{\partial t} C_{ijkl}(t, s)\right| ds + \int_{\tau}^{\tau+\lambda} \left|\frac{\partial}{\partial s} C_{ijkl}(s, \tau)\right| ds \leqslant \varphi_5(\lambda)$$

$$\varphi_4(\lambda) \to 0, \qquad \text{if} \quad \lambda \to 0 \qquad 0 \leqslant \tau \leqslant \tau + \lambda \leqslant t \leqslant T$$

The correspondence between the notations in (1.1) and (2.2) is indicated below. In this case the space H_1 is the space $L_2(\Omega) \times L_2(\Omega) \times L_2(\Omega)$. The operator A_1 is determined from the equality

$$(\mathbf{A}_1\omega \cdot \delta\omega) = \int_\Omega \{T_{ijy}(e_{kl})\,\delta e_{ij} + M_{ijy}(\varkappa_{kl})\,\delta\varkappa_{ij}\} A_1 A_2\, d\alpha_1\, d\alpha_2$$

Here $T_{ijy}(e_{kl})$ means that only the part linear in ω must be taken in the expression T_{ijy} in (2.1). As in [3], it can be shown that the corresponding space H_2 is the subspace $\mathbf{W} = W_2^1(\Omega) \times W_2^1(\Omega) \times W_2^2(\Omega)$, where the norms of H_2 and $\mathbf{W}$ are equivalent on H_2

$$\Phi = \frac{1}{2}\int_\Omega \{T_{ijy}\varepsilon_{ij} + M_{ijy}\varkappa_{ij}\} A_1 A_2\, d\alpha_1\, d\alpha_2$$

The viscous terms in (2.2) correspond to the operator $\mathbf{B}^t$.

Definition 2.1. The vector function $\omega \in H_3(0, T)$ satisfying (2.2) for any vector function $\delta\omega \in D^\circ$ and the first of the initial conditions (2.4) in the sense of (1.5) is called a general solution of (2.2) with the boundary and initial conditions (2.3), (2.4).

Theorem 2.1. Let the Conditions (a) – (d) be satisfied. In this case the problem of the vibrations of a viscoelastic shallow (and nonshallow) shell has at least one generalized solution in the sense of the Definition 2.1, if

$$\mathbf{h} \in H_1, \quad \mathbf{g} \in H_2, \quad \int_0^T \|\mathbf{F}\|_1^2 \, dt < \infty$$

The proof of Theorem 2.1 consists of verifying all the conditions of the abstract Theorem 1.1 upon compliance with the conditions of Theorem 2.1. Conditions (1) – (7) of Theorem 1.1 are carried over word-for-word from [1] and are verified as in [1]. It must just be noted that the functions ψ_1, ψ_2 should perform the role of w_x, w_y in both the shallow and nonshallow shell cases when verifying the sixth condition (see [1], p. 780). The validity of the remaining conditions, except Conditions (9) and (12), is established by the same methods as the validity of their similar parts of Conditions (1)–(7) of Theorem 1.1.

The first part of Condition (9) of Theorem 1.1 relative to the form of the operator $\mathbf{C}^t$ follows from the form of the operator integrated with respect to the time t. The appropriate estimates (1.4) must just be verified. As an illustration, the characteristic term of the left side of the first inequality (1.4) is estimated $(\lambda_k = t_k - t_{k-1})$

$$\left| \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t} \int_\Omega \left\{ \int_{t_{k-1}}^{\tau} C(\tau, \theta)\, \varepsilon(\theta)\, d\theta \frac{\partial \varepsilon(\tau)}{\partial \tau} \right\} d\Omega\, d\tau\, dt \right| =$$

$$\left| - \int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t} \int_\Omega C(\tau, \tau)\, \varepsilon^2(\tau)\, d\Omega\, d\tau\, dt - \right.$$

$$\int_{t_{k-1}}^{t_k} \int_{t_{k-1}}^{t} \int_\Omega \int_{t_{k-1}}^{\tau} C_\tau(\tau, \theta)\, \varepsilon(\theta)\, d\theta \varepsilon(\tau)\, d\Omega\, d\tau\, dt +$$

$$\left. \int_{t_{k-1}}^{t_k} \int_\Omega \int_{t_{k-1}}^{t} C(t, \theta)\, \varepsilon(\theta)\, d\theta \varepsilon(t)\, d\Omega\, dt \right| \leqslant$$

$$\left\{ m_8 \lambda_k + \frac{1}{2} \varphi_4(\lambda_k) + \frac{1}{2} \lambda_k \varphi_5(\lambda_k) \right\} \int_{t_{k-1}}^{t_k} \int_\Omega \varepsilon^2\, d\Omega\, d\tau$$

Integration by parts, interchange of the order of integration, elementary integral inequalities and Condition (d) of Theorem 2.1 were used in the computations. Positive-definiteness of the functional Φ relative to the variables ε_{ij}, $\varkappa_{ij}$ results from Condition (c) of Theorem 2.1. A corollary of this fact and Condition (d) is the first estimate of (1.4) if λ_k is sufficiently small.

Just as Condition (12) of Theorem 1.1, the second estimate of (1.4) is verified analogously.

Theorem 1.2 is carried over directly to the case of a viscoelastic shell.

T h e o r e m 2.2. Let all the conditions of Theorem 2.1 be satisfied and let χ_l be some system of vector functions complete in H_2 and orthonormal in H_1. In this case, the system of differential equations of the Bubnov-Galerkin method (constructed analogously to the system (1.6), (1.7)) has at least one solution in the whole segment $[0, T]$ in each approximation. The set of approximate solutions is weakly compact in $H_3(0, T)$ and each of its limit points is a generalized solution of (2.2) in the sense of Definition 2.1.

Note 1. In the shallow theory case, equations in which the influence of the inertia of longitudinal shell motion is neglected can be considered, i.e. the terms ρu_{itt}, $i = 1, 2$ are missing in (2.2). Equations (2.2) separate naturally into a linear system of equations in u_1, u_2 and still another equation. From this system which is the plane problem of linear quasi-static viscoelasticity in the functions u_1, u_2 in curvilinear coordinates, the displacements u_1, u_2 are found in terms of u_3 by a functional method and are substituted into a new equation. The generalized solution concept is introduced analogously to [1]. Theorems 2.1, 2.2 turn out to be valid for these equations, but morover,the following uniqueness theorem is satisfied.

Theorem. Let all the conditions of Theorem 2.1 be satisfied. In this case the generalized solution of the vibrations equations of a shallow viscoelastic shell, written without taking account of the inertia of the longitudinal shell motions, is unique in the class of functions from $H_3{}^*(0, T)$ (the corresponding norm in $H_2{}^*$ is just the energy of elastic shell bending) for which

$$\| u_{3\alpha_i\alpha_j} \|_{L_{2+\varepsilon}(\Omega)}, \quad \varepsilon > 0, \quad i, j = 1, 2$$

are finite for all $0 \leqslant t \leqslant T$ if the shell characteristics (the middle surface, the outline Γ, the thickness h) are sufficiently smooth and if $\| F_i \|_{L_{2+\varepsilon(\Omega)}}$, $\varepsilon > 0$, $i = 1, 2$ are finite for all $0 \leqslant t \leqslant T$.

Note 2. All the results obtained above are valid even in the particular case, the case of elastic shell vibrations.

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Translated by M. D. F.